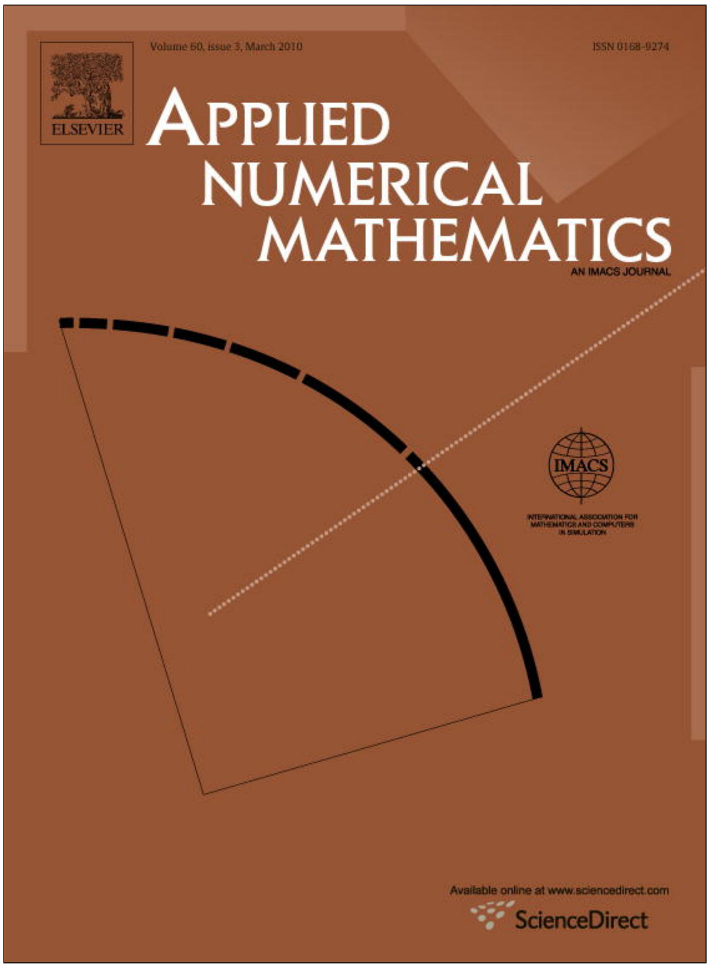
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Notes on accuracy of finite-volume discretization schemes on irregular grids

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ABSTRACT

These notes rebut some overreaching conclusions of Svärd et al., 2008 [19] concerning relations between truncation and discretization errors on irregular grids. Convergence of truncation errors severely degrades on general irregular grids. Such degradation does not necessarily imply a less than design-order convergence of discretization errors.

These notes are a response to the recently published article [19]. The article applies a truncation-error analysis to evaluate accuracy of finite-volume discretization (FVD) schemes on general unstructured grids. The analysis is accompanied by computations performed on regular and irregular grids. We consider some of the conclusions overreaching in application to irregular-grid computations.

On regular grids, convergence of *truncation errors* is an accurate indicator of convergence of discretization errors, provided discrete boundary conditions are adequate. However, the truncation-error convergence is often misleading for FVD schemes defined on irregular (e.g., unstructured) grids. As shown in [19] and twenty years earlier in [18], the second-order convergence of truncation errors for some commonly used FVD schemes can be achieved only on grids with a certain degree of geometric regularity. Other studies, e.g., [2–6,9–15,17,20,21], showed that truncation-error convergence degradation on irregular grids does not necessarily imply a degradation of discretization-error convergence. In [13], discretization schemes in which convergence of discretization errors surpasses the convergence of truncation errors were called *supra-convergent* with references dated back to the 1960s [21].

Plentiful computational evidence and a solid body of theory found in the literature demonstrate that on irregular grids, the design-order discretization-error convergence can be achieved even when truncation errors exhibit a lower-order convergence or, in some cases, do not converge at all. Note that these results do not contradict the Lax theorem, which states that consistency (convergence of truncation errors) and stability are sufficient (not necessary) for convergence of discretization errors. While a rigorous proof of discretization error convergence for FVD schemes on general irregular grids is not yet available, there are several recent publications addressing supra-convergence on irregular grids. Eriksson and Nordström [9] analyze one-dimensional (1D) elliptic equations on irregular grids with centered and randomly shifted locations of the dual grid points (flux locations) and prove the discretization-error convergence of orders 2 and 1.5, respectively. Barbeiro [2]

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proves second-order convergence of discretization errors for formally inconsistent (no truncation-error convergence) discretizations of two-dimensional (2D) elliptic equations on nonuniform grids. Papers [6,17] consider "inconsistent" schemes for advection equations in 1D and 2D and prove convergence of discretization errors. Although we do not show it here, a rigorous proof is in hand for the design-order discretization-error convergence of upwind (and upwind-biased) FVD schemes for constant-coefficient advection equations on random 1D grids. Other discretization-error convergence proofs for some formally inconsistent discretization schemes can be found in Refs. [4,13,21].

Article [19] applied a truncation-error analysis to FVD schemes for the Poisson equation. A "thin-layer" approximation was analyzed. It was shown that the truncation error is $O(1)$ (i.e., does not converge) in grid refinement unless the grids are regular. The discretization error of the scheme was inferred to be non-convergent. By coincidence, the particular thin-layer FVD scheme considered in [19] is indeed zeroth-order accurate even on non-orthogonal structured grids [16]. In [19], a general conclusion was drawn that "a compact finite volume approximation of the Laplacian has to rely on symmetries in the grid to be first-order accurate." This conclusion is incorrect. For example, a common finite-volume scheme equivalent to a Galerkin finite-element approximation (linear elements) on triangles satisfies the definition of a compact scheme and is known to have second-order discretization errors (and zeroth-order truncation errors) on irregular (non-symmetric) grids. FVD schemes for elliptic equations exhibiting similar supra-convergence properties on general mixed-element grids can be found in [8,20].

Article [19] also considered an edge-based *central* FVD scheme for an advection equation on mixed-element and perturbed quadrilateral grids. Truncation-error analysis showed a zeroth-order convergence in the L_∞-norm. Supporting computations showed a zeroth-order convergence of discretization errors. It was concluded that FVD schemes for an advection equation are non-convergent on non-smooth irregular grids. The conclusion is incorrect in general because there are counter examples of FVD schemes with truncation errors that do not converge on general irregular grids but with discrete solutions that converge with at least first order in any norm [8]. The numerical scheme considered in [19] is not representative of current practice—the central scheme is known to exhibit erratic convergence of discretization errors in grid refinement because of lack of h-ellipticity, see, for example [7,8,22]. Note that the article [9] also considers a central scheme for a 1D constant-coefficient advection equation on irregular grids and proves that the mean discretization-error convergence order is at least 0.5, which is better than the zeroth-order convergence predicted in [19] and agrees well with the computational results shown in [8] for a central 2D scheme. For multidimensional advection equations and inviscid compressible and incompressible flow equations, the second-order convergence of discretization errors has been previously demonstrated using upwind edge-based schemes on general simplicial (triangular and tetrahedral) grids; the first-order convergence has been observed on general mixed and perturbed quadrilateral (hexagonal) grids [1,8,20]. The reason for not attaining the design second-order convergence of discretization errors has been traced in [8,20] to the first-order accuracy of control-volume boundary flux integration, which is typical for edge-based FVD schemes on irregular non-simplicial grids.

In summary, degradation of truncation-error convergence does not necessarily imply a lower-order convergence of discretization errors. While the individual computations in [19] appear to be correct, several conclusions derived from a truncation-error analysis regarding degradation of discretization error convergence in irregular-grid computations are overreaching. A vast literature on supra-convergence and substantial computational evidence show that the design-order discretization-error convergence can be achieved even when truncation errors exhibit a lower-order convergence or, in some cases, do not converge at all.

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