



A01-31150

**AIAA 01-2575**

**Factorizable Upwind Schemes: The  
Triangular Unstructured Grid  
Formulation**

**David Sidilkover**

*ICASE*

**Eric J. Nielsen**

*NASA Langley Research Center*

*Hampton, VA*

**15th AIAA Computational Fluid  
Dynamics Conference**  
11-14 June 2001/Anaheim, CA

# Factorizable upwind schemes: the triangular unstructured grid formulation

David Sidilkover\*

ICASE, NASA Langley Research Center, Hampton, VA 23681

Eric J. Nielsen†

NASA Langley Research Center, Hampton, VA 23681

The upwind factorizable schemes for the equations of fluid was introduced recently. They facilitate achieving the Textbook Multigrid Efficiency (TME) and are expected also to result in the solvers of unparalleled robustness. The approach itself is very general. Therefore, it may well become a general framework for the large-scale Computational Fluid Dynamics. In this paper we outline the triangular grid formulation of the factorizable schemes. The derivation is based on the fact that the factorizable schemes can be expressed entirely using vector notation, without explicitly mentioning a particular coordinate frame. We describe the resulting discrete scheme in detail and present some computational results verifying the basic properties of the scheme/solver.

## Introduction

This work is a part of effort going on at NASA Langley for several years towards constructing a new generation of the flow solvers (see, for instance Thomas et al.<sup>1</sup>, Roberts et al.<sup>2</sup>). The key idea, that was suggested by Brandt,<sup>3</sup> is to use a special relaxation that recognizes the mixed character of a system of PDEs. Then each sub-factor of the system can be treated in different (optimal for it) way. It is well-known that the Euler equations “consist” of two different factors: advection and full-potential operators. The advection part can be treated very efficiently, say, by the marching relaxation. The full-potential operator is of the elliptic type in the subsonic regime. Therefore, it can be treated very efficiently by multigrid. In the supersonic regime it becomes hyperbolic: wave equation with respect to the flow direction. For Mach number substantially larger than one, the entire system can be solved efficiently by marching. Nearly sonic speed regime can be dealt with by multigrid, but requires some special care.

Solvers based on such a special (Distributive) relaxation were constructed initially for incompressible flow and were based on the staggered grid discretizations. The optimal multigrid efficiency was demonstrated Brandt and Yavneh.<sup>4</sup> Staggered-grid discretizations exist also for the compressible flow equations. However, they all are limited to the subsonic regime, since they have no shock-capturing capabilities. The standard shock-capturing schemes, on the other hand, are

not factorizable, i.e. they do not reflect the mixed character of the PDEs (see Sidilkover<sup>5</sup>). Therefore, it is not possible to construct a relaxation of Distributive type that can be used with these schemes.

Clearly, there is a need for discretizations that are both factorizable and have shock-capturing capabilities. A factorizable upwind scheme was constructed in Sidilkover<sup>6</sup> for the case of Cartesian grids. A detailed description of its extension to the case of structured body-fitted grids is given in Sidilkover et al.<sup>7</sup> A set of numerical results was presented in Roberts et al.<sup>8</sup>

The purpose of this paper is to present a construction of a factorizable scheme on triangular unstructured grids.

## Euler equations and their properties

The non-conservative quasilinear formulation of the compressible Euler equations in three dimensions can be written using vector notation as follows

$$\vec{u} \cdot \nabla s = 0 \quad (1a)$$

$$\rho \vec{u} \cdot \nabla \vec{u} + \nabla p = 0 \quad (1b)$$

$$\rho c^2 \nabla \cdot \vec{u} + \vec{u} \cdot \nabla p = 0, \quad (1c)$$

where  $s$  denotes the entropy and

$$ds = d\rho - \frac{dp}{c^2}, \quad (2)$$

$\vec{u}$  is the velocity vector, the pressure  $p$  is given by

$$p = (\gamma - 1) \left( e - \rho \frac{|\vec{u}|^2}{2} \right) \quad (3)$$

the speed of sound

$$c = \sqrt{\frac{\gamma p}{\rho}}. \quad (4)$$

\*Senior Staff Scientist

†Research Scientist

Copyright © 2001 by the American Institute of Aeronautics and Astronautics, Inc. No copyright is asserted in the United States under Title 17, U.S. Code. The U.S. Government has a royalty-free license to exercise all rights under the copyright claimed herein for Governmental Purposes. All other rights are reserved by the copyright owner.

In this section we recall some of the basic properties of the equations (1). It is sufficient for the purpose of the analysis to assume constancy of the coefficients. It is known that this system of equations is of the mixed type: it consists of the advection and the Full-Potential factors. This can be made obvious by introducing the new set of variables.

Recall that a vector field can be decomposed into solenoidal and irrotational parts

$$\vec{u} = \nabla \times \vec{\psi} + \nabla \phi, \quad (5)$$

where  $\phi$  is the potential and  $\vec{\psi}$  is the streamfunction. The pressure gradient is related to the gradient of the potential as follows

$$dp = -\rho \vec{u} \cdot \nabla d\phi \quad (6)$$

Substituting the new variables  $\phi$  and  $\vec{\psi}$  into the pressure equation we obtain the Full-Potential equation

$$\rho[c^2 \nabla^2 - (\vec{u} \cdot \nabla)^2] \phi = 0 \quad (7)$$

Note, that all the terms involving the streamfunction cancel out.

Performing the variable substitution in the momentum equations gives

$$\rho \vec{u} \cdot \nabla (\nabla \times \vec{\psi}) = 0 \quad (8)$$

Note, that all the terms involving the potential variable cancel out.

Introducing a new variable – vorticity

$$\vec{\Omega} = \Delta \vec{\psi} \quad (9)$$

and applying operator  $\nabla \times$  to (8), we obtain

$$\rho \vec{u} \cdot \nabla \vec{\Omega} = 0 \quad (10)$$

This verifies indeed that the Euler system is of the mixed type. The advection factor is represented by the equations for entropy (1a) and vorticity (10). The full-potential factor is given by (7). It also makes it clear that (for the linear constant coefficients case) the momentum equations (1b) drive the solenoidal part of the solution, while the irrotational part of the solution is subject solely to the pressure equation (1c).

In a general nonlinear case (away from singularities, like shocks and contact discontinuities) there is a weak coupling between different factors due to the so-called subprincipal terms. This coupling can be neglected for the purpose of the construction of a fast solver (see Brandt<sup>3</sup>). Therefore, the latter can rely entirely on the analysis of the linear case.

## Preparations for the scheme construction

When constructing a discrete approximation to the Euler equations, the central scheme can serve a basic building block. However, it is crucial to include

a certain artificial dissipation in the discretization for stability reasons. One of the additional problems than becomes how to compensate for the loss of accuracy due to the artificial dissipation. Various ways to deal with this issue received an extensive coverage in the literature. Constructing a *factorizable* scheme implies resolving this issue in a very specific way.

### FDA analysis

The First Differential Approximation (FDA) (or the *modified equations*) corresponding to a certain discrete scheme is the PDEs augmented by the leading error terms.

We shall start our analysis with formulating the FDA for the factorizable genuinely multidimensional scheme. The observation made in regarding the genuinely multidimensional upwind scheme introduced in Sidilkover<sup>9</sup> was that a part of the artificial dissipation present in the discrete momentum equations (in subsonic case) is proportional to the gradient of the residual of the pressure equation. The artificial dissipation of the pressure equation is proportional to the divergence of the residuals of the momentum equations. A vector formulation of the entire scheme (on the Cartesian grids) is given in Sidilkover.<sup>10</sup>

The fact that the entire scheme can be expressed using the vector notation appeared to be very instrumental for the purpose of extending the factorizable to the structured body-fitted grids (See Sidilkover et al<sup>7</sup> and Roberts et al<sup>8</sup>). It is of very important for the purpose of this paper too.

The FDA of a factorizable scheme for the Euler equations is given by the following

$$\mathbf{q}s = 0 \quad (11a)$$

$$\rho \mathbf{q} \vec{u} + \nabla p - \frac{\sigma_m l}{2} \frac{1}{c} \nabla (\rho c^2 \nabla \cdot \vec{u} + \vec{u} \cdot \nabla p) - \nabla D = 0 \quad (11b)$$

$$\rho c^2 \nabla \cdot \vec{u} + \vec{u} \cdot \nabla p - \frac{\sigma_p l}{2} c \nabla \cdot (\rho \vec{u} \cdot \nabla \vec{u} + \nabla p) = 0 \quad (11c)$$

The term  $\nabla D$  in the momentum equations plays an important role when the operator  $\mathbf{q}$  is discretized using an advection scheme of a certain type to maintain the second order accuracy and factorizability of the whole Euler scheme. This special type of the advection scheme allows to upgrade the accuracy of the advection factor to the second order without affecting the discrete full-potential part. For now we omit the term  $\nabla D$  and consider  $\mathbf{q}$  to correspond to the standard first order accurate advection scheme.

A factorizable scheme corresponding to the FDA as given by (11) is stable for subsonic case only. The scheme can be extended so it will be valid for transonic/supersonic regime by a simple modification: in-

roduction of a cut-off parameter on the terms involving pressure in the artificial dissipation

$$\mathbf{qs} = 0 \quad (12a)$$

$$\rho \mathbf{q} \bar{u} + \nabla p - \frac{\sigma_m l}{2} \frac{1}{c} \nabla (\rho c^2 \nabla \cdot \bar{u} + \frac{1}{\kappa} \bar{u} \cdot \nabla p) - \nabla D = 0 \quad (12b)$$

$$\rho c^2 \nabla \cdot \bar{u} + \bar{u} \cdot \nabla p - \frac{\sigma_p l}{2} c \nabla \cdot (\rho \bar{u} \cdot \nabla \bar{u} + \kappa \nabla p) = 0, \quad (12c)$$

where

$$\kappa = \max(1, M^2) \quad (13)$$

We shall demonstrate the factorizability property of FDA corresponding to the new scheme. Introduce the auxiliary variables – potential  $\phi$  and streamfunction  $\bar{\psi}$  and substitute the following expressions for the variables  $\bar{u}, p$

$$\bar{u} = (I - \frac{\sigma_p l}{2} \frac{1}{c} \bar{u} \cdot \nabla) \nabla \times \bar{\psi} + \nabla (I - \frac{\sigma_m l}{2} \frac{1}{c} \bar{u} \cdot \nabla) \phi \quad (14a)$$

$$dp = -\rho (\bar{u} \cdot \nabla - \frac{\sigma_m l}{2} c \nabla^2) d\phi \quad (14b)$$

into (11). Introducing the vorticity variable

$$\bar{\Omega} = \nabla \times (I - \frac{\sigma_p l}{2} \frac{1}{c} \bar{u} \cdot \nabla) \nabla \times \bar{\psi} \quad (15)$$

and applying the  $\nabla \times$  operator to the momentum equations we obtain

$$\rho \bar{u} \cdot \nabla \bar{\Omega} = 0. \quad (16)$$

Note, that, as well as in the PDE case, all the terms involving the potential variable canceled out. Substituting the auxiliary variables into the pressure equation, it is easy to verify that all the terms involving the streamfunction cancel.

We can summarize that due to the specific form of the artificial dissipation, the FDA (11) of the discrete scheme is factorizable – it reflects the mixed character of the original system of PDEs.

### Special discrete operators

We consider the two-dimensional case from now on. When discretizing the derivatives, a special care needs to be taken of what kind of discrete operators are used in order to preserve the factorizability property at the discrete level.

Note, that when demonstrating the factorizability property of the scheme's FDA (11) we used the facts of the following type

$$\begin{aligned} \partial_{xx} \partial_y &= \partial_{xy} \partial_x \\ \partial_{yy} \partial_x &= \partial_{xy} \partial_y \\ \partial_{xy} \partial_{xy} &= \partial_{xx} \partial_{yy} \end{aligned} \quad (17)$$

In order to obtain the factorizable discrete scheme, we need to introduce some finite differences that possess the property analogous to (17). Such finite differences were introduced for the case of structured grids in.<sup>6</sup>

We have to find such differences for the case of triangular grids (see Fig1), where  $(x, y)$  is a local (non-orthogonal) frame. We shall illustrate this on a simple example. Consider approximating the partial derivative  $\partial_{xx}$ . Assuming that we have at our disposal the “compact” stencil that involves 7 points: 0, 1, 2, 3, 4, 5, 6, there is only one way of doing it, namely by  $\bar{\partial}_{xx}^h$  defined as follows

$$\bar{\partial}_{xx}^h u = (u_4 - 2u_0 + u_1)/h^2 \quad (18)$$

Differences defined in such a way do not have the property (17). We can conclude that using the compact stencil 7-point stencil only there is no way to achieve this property. We know from<sup>6</sup> that the 9-point box structured grid stencil is sufficient for this purpose. There are several ways to augment the compact 7-point stencil to the 9-point one in the current triangular grid context. It is clear that the 7-point stencil, therefore, needs to be augmented. We can do it by adding to it 6 more nodes: 7, 8, 9, 10, 11, 12. The partial derivative  $\partial_{xx}$  can then be approximated by a *wide* difference

$$\bar{\delta}_{xx}^h = \frac{[(u_4 - 2u_0 + u_1)/2 + (u_{10} + u_3 - u_2 - u_8)/8 + (u_{11} + u_6 - u_6 - u_7)/8]/h^2}{1} \quad (19)$$

The derivatives  $\partial_{yy}, \partial_x, \partial_y$  can be approximated in the analogous way. It is easy to verify that such differences possess the property (17).

### Structure of some artificial dissipation terms

The central part of the scheme is the constructed in the standard way. The artificial dissipation terms corresponding to the advection scheme are evaluated in the standard fashion as well. A special care needs to be taken of the other artificial dissipation terms.

Recall, that the artificial dissipation terms in the momentum equations (11b) that are subject to the gradient operator are the residual of the pressure equation. Denote them

$$R_p = \rho c^2 \nabla \cdot \bar{u} + \bar{u} \cdot \nabla p \quad (20)$$

and the expression subject to the action of the divergence operator in the pressure equation (11c) is the residual of the momentum equations

$$\bar{R}_m = \rho \bar{u} \cdot \nabla \bar{u} + \nabla p \quad (21)$$

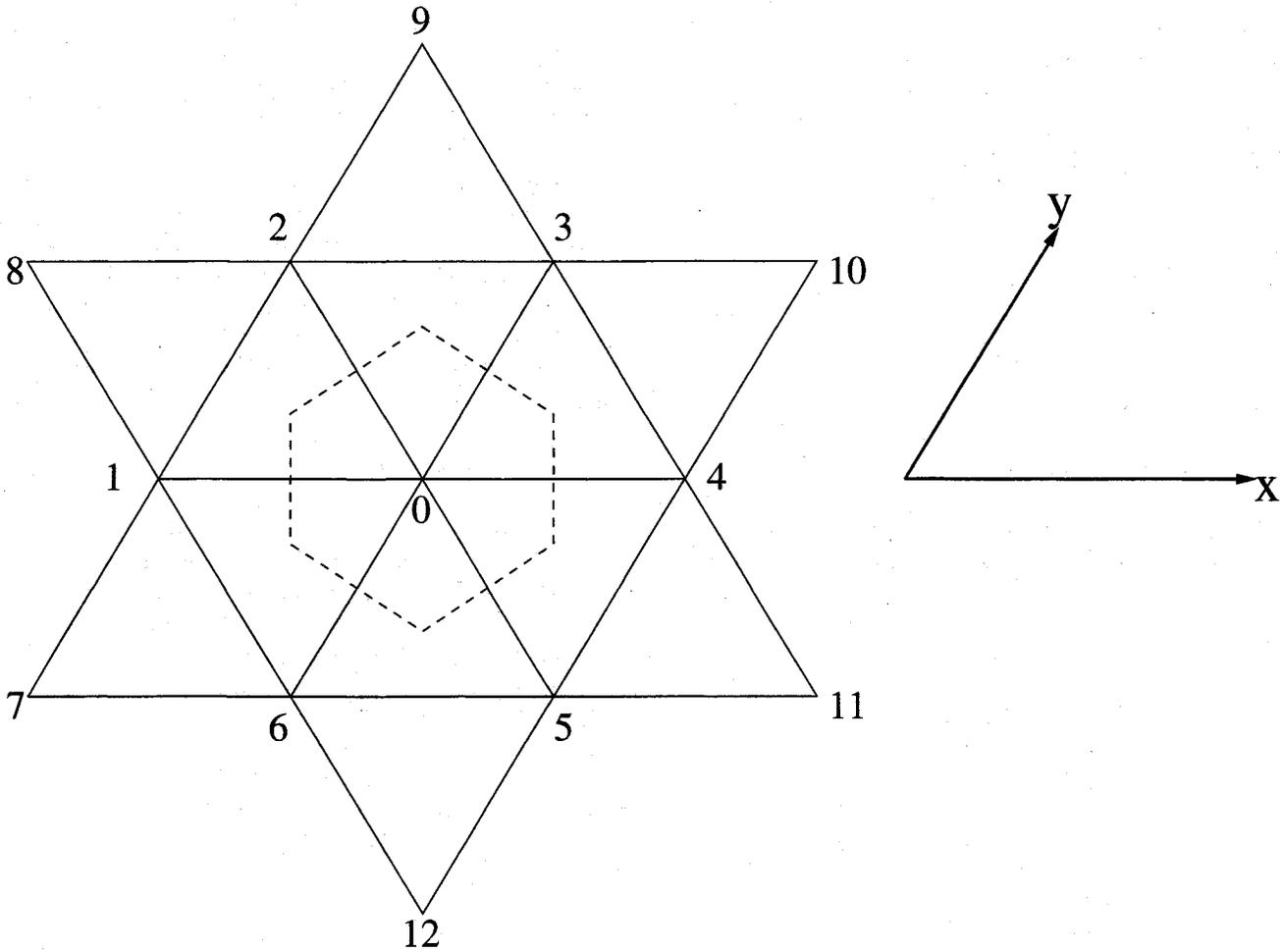


Fig. 1 Computational grid segment and a control volume.

### Constructing the discrete scheme

Our goal now is to derive a discretization of a conservation law (scalar and a system). For this purpose we need to evaluate the numerical fluxes through each of the faces of the dual-median cell (see Fig.1).

#### Global and local coordinate frames

$$H = \begin{pmatrix} \alpha_\xi & \alpha_\eta \\ \beta_\xi & \beta_\eta \end{pmatrix}, \quad (22)$$

where  $(\alpha_\xi, \beta_\xi)$  and  $(\alpha_\eta, \beta_\eta)$  are the unit vectors in the direction of the  $\xi$  and  $\eta$  coordinate axes respectively. The relationship between the Cartesian and contravariant velocity components is described as follows

$$H \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \quad (23)$$

or

$$H^{-1} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}. \quad (24)$$

The Jacobian of this coordinate rotation

$$J \equiv \det H = \alpha_\xi \beta_\eta - \beta_\xi \alpha_\eta \quad (25)$$

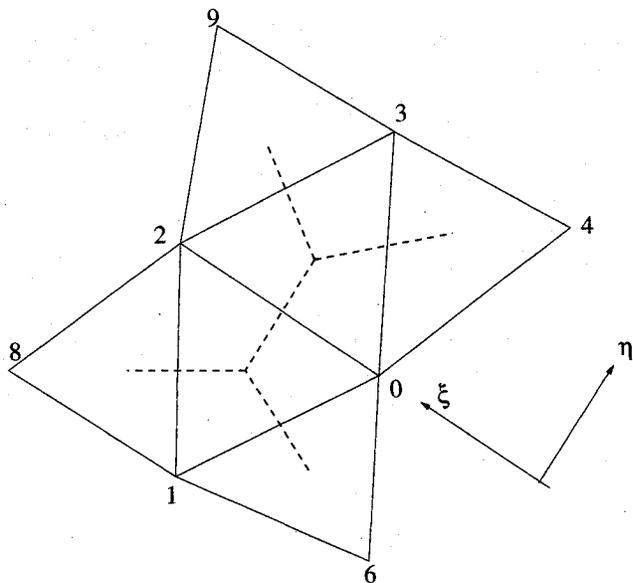


Fig. 2 Nodes used for evaluating the flux through a face of dual-median cell.

The inverse of the Hessian  $H$

$$H^{-1} = \frac{1}{J} \begin{pmatrix} \beta_\eta & -\alpha_\eta \\ -\beta_\xi & \alpha_\xi \end{pmatrix} \quad (26)$$

It is convenient to use the scaled contravariant velocity components

$$\begin{aligned} \bar{U} &\equiv J\bar{u} = u\beta_\eta - v\alpha_\eta \\ \bar{V} &\equiv J\bar{v} = -u\beta_\xi + v\alpha_\xi \end{aligned} \quad (27)$$

The relationship between the Cartesian and covariant velocity components is given by the following

$$H^T \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \bar{U} \\ \bar{V} \end{pmatrix} \quad (28)$$

or

$$\begin{aligned} \bar{U} &= u\alpha_\xi + v\beta_\xi \\ \bar{V} &= u\alpha_\eta + v\beta_\eta \end{aligned} \quad (29)$$

The covariant and contravariant velocities are related as follows

$$H^T H \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} \bar{U} \\ \bar{V} \end{pmatrix} \quad (30)$$

$$H^T H = \begin{pmatrix} \alpha_\xi^2 + \beta_\xi^2 & \alpha_\xi\alpha_\eta + \beta_\xi\beta_\eta \\ \alpha_\xi\alpha_\eta + \beta_\xi\beta_\eta & \alpha_\eta^2 + \beta_\eta^2 \end{pmatrix} \quad (31)$$

The total velocity squared

$$|\bar{u}|^2 = u^2 + v^2 = \bar{u}\bar{U} + \bar{v}\bar{V} \quad (32)$$

### Scalar advection

Consider a scalar advection equation

$$s_t + us_x + vs_y = 0. \quad (33)$$

The discrete equation to solve for  $s$  at point 0 is obtained by balancing fluxes through the surface of the dual median cell (see Fig.1)

$$\sum_{i=1}^{N_d} [f h_\eta]_i = 0, \quad (34)$$

where  $h_\eta$  is the length of the corresponding face and the numerical flux

$$f = -\frac{1}{2}h_\xi|\bar{U}|\partial_\xi^h s + \frac{1}{2}\bar{U}(s_0 + s_2) \quad (35)$$

where  $\partial_\xi^h$  stands for a divided difference

$$h_\xi\partial_\xi^h s = s_2 - s_0 \quad (36)$$

The discrete equation to solve for  $s$  at point 0 is obtained by substituting numerical fluxes evaluated by analogy to (35) on all the faces of the dual-median cell and substituting them into the flux-balance equation (34).

### Euler system

Integrating the Euler equations in the conservative form over the control-volume (dual-median cell) and applying the Green's theorem, we obtain

$$\oint_{\partial C} \mathbf{F} \cdot \bar{\mathbf{n}} dl = 0, \quad (37)$$

where  $\partial C$  is the control volume's boundary  $\bar{\mathbf{n}}$  is a unit vector normal to the boundary and

$$\mathbf{F} = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (E+p)u \end{pmatrix} \hat{i} + \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (E+p)v \end{pmatrix} \hat{j} \quad (38)$$

A conservative discretization of the Euler system can be written in the following form

$$\sum_{i=1}^{N_d} [\bar{\mathbf{F}} h_\eta]_i = 0. \quad (39)$$

The numerical flux through a face can be represented as a sum of central and the artificial dissipation parts

$$\bar{\mathbf{F}} = \bar{\mathbf{F}}^c + \bar{\mathbf{F}}^d \quad (40)$$

The central portion of the flux is given by

$$\bar{\mathbf{F}}^c = \frac{1}{2}[\mathbf{F}(\mathbf{u}_L) + \mathbf{F}(\mathbf{u}_R)] \cdot \bar{\mathbf{n}}, \quad (41)$$

where now  $\bar{\mathbf{n}} = (\beta_\eta, -\alpha_\eta)$  is a unit vector normal to the face. The rest of this section is dedicated to the question of deriving the diffusive portion of the numerical fluxes.

We would like to emphasize that it is fairly simple to implement the new discretization within the existing control-volume computer codes. It requires only the new numerical flux routine. Such a routine can be written in several simple steps, starting from the standard upwind scheme and performing the modifications gradually.

#### The standard upwind scheme

The first step is to rewrite the standard upwind for the subsonic case without explicit mention of the characteristic variables, etc. The scheme is given by the following numerical flux

$$\bar{\mathbf{f}}^d = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = -\frac{1}{2}h_\xi \begin{pmatrix} |\bar{U}|\partial_\xi^h s \\ \rho c \partial_\xi^h \bar{U} + \bar{U}/c \partial_\xi^h p \\ \rho |\bar{U}|\partial_\xi^h \bar{V} \\ \rho c \bar{U} \partial_\xi^h \bar{U} + c \partial_\xi^h p \end{pmatrix} \quad (42)$$

The derivation presumes that these fluxes correspond to the local orthogonal coordinate frame associated with the cell-face. Therefore, the momentum equations diffusive fluxes can be rotated to the global coordinate frame  $(x, y)$  by in the following way

$$\begin{pmatrix} f_2 \\ f_3 \end{pmatrix} := \begin{pmatrix} \beta_\eta & \alpha_\eta \\ -\alpha_\eta & \beta_\eta \end{pmatrix} \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \quad (43)$$

The obtain diffusive fluxes correspond to the quasi-linear nonconservative formulation of the Euler equations. The need to be transformed, therefore, to the form appropriate for the conservative discretization

$$\bar{F}^d = M \bar{f}^d \quad (44)$$

where

$$M = \begin{pmatrix} 1 & 0 & 0 & 1/c^2 \\ u & 1 & 0 & u/c^2 \\ v & 0 & 1 & v/c^2 \\ (\bar{u} \cdot \bar{u})/2 & u & v & 1/(\gamma - 1) + (\bar{u} \cdot \bar{u})/(2c^2) \end{pmatrix} \quad (45)$$

#### A modified scheme

As an intermediate step towards constructing the factorizable scheme we can consider the following case:

$$\bar{f}^d = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = -\frac{1}{2} h_\xi \begin{pmatrix} |\bar{U}| \partial_\xi^h s \\ \rho c \partial_\xi^h \bar{U} + \bar{U} / c \partial_\xi^h p \\ \rho |\bar{U}| \partial_\xi^h \bar{V} \\ \rho c \bar{U} \partial_\xi^h \bar{U} + c \partial_\xi^h p \end{pmatrix} \quad (46)$$

The main difference from the standard scheme is that the momentum equations diffusive fluxes (the second and third components) are now attributed to the momentum equations in the covariant directions ( $\xi$  and  $\eta$ ). Therefore, the transformation back to the global orthogonal frame takes the following form

$$\begin{pmatrix} f_2 \\ f_3 \end{pmatrix} := (H^T)^{-1} \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \quad (47)$$

This change is necessary towards eventually obtaining the approximation for the gradient operator term in (11b)

#### Another intermediate step

$$\bar{f}^d = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = -\frac{1}{2} h_\xi \begin{pmatrix} |\bar{U}| \partial_\xi^h s \\ \rho c \partial_\xi^h \bar{U} + \bar{U} / c \partial_\xi^h p \\ \rho |\bar{U}| \partial_\xi^h \bar{V} \\ \rho c \bar{U} \partial_\xi^h \bar{U} + \frac{1}{J} c \partial_\xi^h p \end{pmatrix} \quad (48)$$

The coefficient in front of the pressure difference in the fourth component constitutes the change from the previous case. It is necessary in order arrive later to the approximation of the divergence term

#### The factorizable scheme

Now we shall incorporate the correction (mixed derivative) terms into the scheme, so that some of the terms in the artificial dissipation can be viewed as residuals of the momentum and the pressure equations and, therefore, are second order small. The latter modification together with introducing the *wide* differences that possess the commutativity properties results in a

*factorizable* scheme. Implementation of all these steps can be done in several steps as well, testing the routine after each modification.

We start from re-interpreting of some standard notions. A standard “narrow” *divided* difference can be defined also as

$$\partial_\xi^h = \frac{S^{012} \partial_\xi^{012} + S^{023} \partial_\xi^{023}}{S^{012} + S^{023}} \quad (49)$$

Define a “wide *divided* difference”

$$\bar{\partial}_\xi^h = \frac{(2S^{012} \partial_\xi^{012} + 2S^{023} \partial_\xi^{023} + S^{061} \partial_\xi^{061} + S^{182} \partial_\xi^{182} + S^{293} \partial_\xi^{293} + S^{034} \partial_\xi^{034}) / (2S^{012} + 2S^{023} + S^{061} + S^{182} + S^{293} + S^{034})}{(2S^{012} + 2S^{023} + S^{061} + S^{182} + S^{293} + S^{034})} \quad (50)$$

Similarly, we can define  $\partial_\eta^h$  and  $\bar{\partial}_\eta^h$ . Adding the some specific  $\eta$ -derivative terms (all the differences used here are *narrow*) to the artificial dissipation of the pressure equation obtain the residual of the momentum equation in the direction normal to the cell face

$$\mathbb{R}_U^h = \rho (\bar{U} \partial_\xi^h + \bar{V} \partial_\eta^h) \bar{U} + \frac{1}{J} [\partial_\xi^h + (\alpha_\xi \alpha_\eta + \beta_\xi \beta_\eta) \partial_\eta^h] p \quad (51)$$

We also add some  $\eta$ -derivative terms to the artificial dissipation of the momentum equation in  $\xi$  direction to obtain the residual of the pressure equation. All the differences used here are *wide*. The notation ( $\mathbb{R}$  for the residual) reflects this fact

$$\mathbb{R}_p^h = \rho c^2 (\bar{\partial}_\xi^h \bar{U} + \bar{\partial}_\eta^h \bar{V}) + (\bar{U} \bar{\partial}_\xi^h + \bar{V} \bar{\partial}_\eta^h) p \quad (52)$$

The artificial dissipation then takes the following form

$$\bar{f}^d = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} h_\xi |\bar{U}| \partial_\xi^h s \\ \sigma_m l \frac{1}{c} \mathbb{R}_p^h + h_\xi \rho |\bar{U}| \partial_\xi^h \bar{U} \\ h_\xi \rho |\bar{U}| \partial_\xi^h \bar{V} \\ \sigma_p l c \mathbb{R}_U^h \end{pmatrix} \quad (53)$$

The need for rescaling the artificial dissipation in order to avoid the quasi-ellipticity of the full-potential factors approximation for the low-speed flow regions was established in.<sup>6</sup> The scaling parameters  $\sigma_m, \sigma_p$  serve this purpose. In the conducted preliminary numerical test (see next section) they were taken to be 1. The parameter  $l$  was taken to be equal to  $h_\xi$ .

There remains a need to introduce some modifications into the central part of the scheme. It is necessary for *factorizability* that the pressure gradient term in the momentum equations is approximated by *differences*. Using *wide* differences to approximate the pressure advection operator in the pressure equation is not necessary for factorizability but is still beneficial since it results in a better form of the discrete

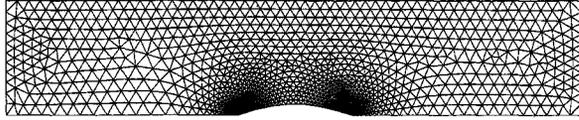


Fig. 3 The computational grid.

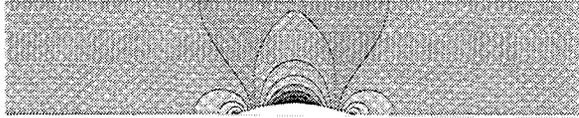


Fig. 4 Solution: density contours.

full-potential factor. These modifications can be introduced in a very simple way using a trick by Tom Roberts: adding certain terms to the artificial dissipation. Introduce the following *undivided* difference

$$\bar{\delta}_{\eta\eta}^h = \frac{[(2S^{023}\partial_{\eta}^{023} + S^{293}\partial_{\eta}^{293} + S^{034}\partial_{\eta}^{034}) - (2S^{012}\partial_{\eta}^{012} + S^{061}\partial_{\eta}^{061} + S^{182}\partial_{\eta}^{182})]}{2h_{\xi}} \quad (54)$$

The artificial dissipation terms are then augmented as follows

$$\begin{pmatrix} f_2 \\ f_4 \end{pmatrix} := \begin{pmatrix} f_2 + \bar{\delta}_{\eta\eta}^h p \\ f_4 + \bar{U} \bar{\delta}_{\eta\eta}^h p \end{pmatrix} \quad (55)$$

Returning to the global Cartesian coordinate frame

$$\begin{pmatrix} f_2 \\ f_3 \end{pmatrix} := (H^T)^{-1} \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \quad (56)$$

and converting to the conservative form

$$\vec{F}^d = M \vec{f}^d \quad (57)$$

This describes the scheme that was used in the preliminary numerical experiment reported below.

### Preliminary numerical results

The work on implementing the new scheme within the FUN2D<sup>11</sup> code has just began. Our very first aim is just to implement the numerical fluxes and to verify the correctness of the residual evaluation.

The testcase presented is a subsonic flow (Mach = .2) in a channel with a bump. The grid consists of 1375 nodes (see Fig.3). A second order version of the new scheme was used. The contour plots of density are presented in Fig.4. We also present for comparison in Fig.5 density contours of the solution to the same problem using the standard second order scheme with limiters switched off. order upwind scheme. The solution obtained using the new scheme is at least as accurate as the one using the standard scheme.

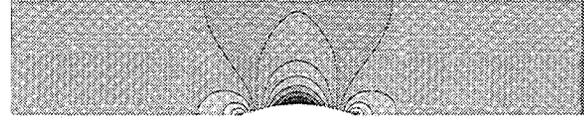


Fig. 5 Solution obtained using the second order upwind scheme: density contours.

### References

- <sup>1</sup>Thomas, J. L., Diskin, B., and Brandt, A., "Distributed relaxation multigrid and defect correction applied to the compressible Navier-Stokes equations," AIAA 99-3334, June 2000, AIAA CFD meeting, Norfolk, VA, 1999.
- <sup>2</sup>Roberts, T. W., Swanson, R. C., and Sidilkover, D., "An algorithm for ideal multigrid convergence for the the steady Euler equations," 1999.
- <sup>3</sup>Brandt, A., "Multigrid techniques: 1984 Guide with applications to fluid dynamics," The Weizmann Institute of Science, Rehovot, Israel, 1984.
- <sup>4</sup>Brandt, A. and Yavneh, I., "Accelerated multigrid convergence and high Reynolds recirculating flows," *SIAM J. Sci. Statist. Comput.*, Vol. 14, 1993, pp. 607-626.
- <sup>5</sup>Sidilkover, D., "Some approaches towards constructing optimally efficient multigrid solvers for the inviscid flow equations," *Computers & Fluids*, Vol. 28, 1999, pp. 551-571.
- <sup>6</sup>Sidilkover, D., "Factorizable schemes for the equations of fluid flow," Report No. 99-20, ICASE, 1999, Submitted for publication.
- <sup>7</sup>Sidilkover, D., Roberts, T. W., and Thomas, J. L., "The generalized coordinate formulation of the factorizable upwind scheme," 2000, In preparation.
- <sup>8</sup>Roberts, T. W., Sidilkover, D., and Thomas, J. L., "Multigrid relaxation of a factorizable conservative discretization of the compressible flow equations," AIAA 00-2252, June 19-22, 2000, AIAA Fluids meeting, Denver, CO.
- <sup>9</sup>Sidilkover, D., "A genuinely multidimensional upwind scheme and efficient multigrid solver for the compressible Euler equations," Report No. 94-84, ICASE, 1994.
- <sup>10</sup>Sidilkover, D., "The factorizable upwind scheme in two and three dimensions," 2000, In preparation.
- <sup>11</sup>Anderson, W. K. and Bonhaus, L. D., "An implicit Upwind Algorithm for Computing Turbulent Flows on Unstructured Grids," *Comput. & Fluids*, Vol. 23, 1994, pp. 1-21.