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Short note

Entropy stable discontinuous interfaces coupling for the three-dimensional compressible Navier–Stokes equations

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1. Introduction

Non-linear entropy stability and a summation-by-parts (SBP) framework are used to derive entropy stable interior interface coupling for the semi-discretized three-dimensional (3D) compressible Navier–Stokes equations. A complete semi-discrete entropy estimate for the interior domain is achieved combining a discontinuous entropy conservative operator of any order [1,2] with an entropy stable coupling condition for the inviscid terms, and a local discontinuous Galerkin (LDG) approach with an interior penalty (IP) procedure for the viscous terms. The viscous penalty contributions scale with the inverse of the Reynolds number (*Re*) so that for $Re \to \infty$ their contributions vanish and only the entropy stable inviscid interface penalty term is recovered. This paper extends the interface couplings presented [1,2] and provides a simple and automatic way to compute the magnitude of the viscous IP term. The approach presented herein is compatible with any diagonal norm summation-by-parts (SBP) spatial operator, including finite element, finite volume, finite difference schemes and the class of high-order accurate methods which include the large family of discontinuous Galerkin discretizations and flux reconstruction schemes.

This note relies on the formalism introduced in [1,3] and complements the new class of interior entropy stable SBP operators of any order for the 3D compressible Navier–Stokes equations on unstructured grids that was proposed in [1,2]. To keep the notation as simple as possible, a uniform Cartesian grid is considered in the derivation. However, the extension to generalized curvilinear coordinates and unstructured grids follows immediately if the transformation from computational to physical space preserves the semi-discrete geometric conservation [4].

The proposed interface coupling technique has been successfully combined with a high order entropy stable discretization for the simulation of two-dimensional (2D) and 3D viscous subsonic and supersonic flows presented in [1,5,6].

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2. Compressible Navier–Stokes equations and entropy function

Consider a fluid in a domain Ω with boundary surface denoted by $\partial\Omega$, without radiation and external volume forces. In this context, the compressible Navier–Stokes equations, equipped with suitable boundary and initial conditions, may be expressed in the form

$$\begin{aligned}
&\frac{\partial q}{\partial t}+\frac{\partial f_i^{(I)}}{\partial x_i}=\frac{\partial f_i^{(V)}}{\partial x_i}, \quad x\in\Omega, \quad t\in[0,\infty),\\
&q|_{\partial\Omega}=g^{(B)}(x,t), \quad x\in\partial\Omega, \quad t\in[0,\infty),\\
&q(x,0)=g^{(0)}(x), \quad x\in\Omega,
\end{aligned} \tag{1}$$

where the Cartesian coordinates, $x=(x_1,x_2,x_3)^\top$, and time, t, are the independent variables. Note that in (1) Einstein notation is used. The vectors $q(x,t)$, $f_i^{(I)}=f_i^{(I)}(q)$, and $f_i^{(V)}=f_i^{(V)}(q,\nabla q)$ are the conserved variables, and the inviscid and viscous fluxes in the i direction, respectively.[1] Both boundary conditions, $g^{(B)}$, and initial data, $g^{(0)}$, are assumed to be bounded, $L^2\cap L^\infty$. Furthermore, $g^{(B)}$ is also assumed to contain linearly well-posed data. The conservative variable vector is $q=(\rho,\rho u_1,\rho u_2,\rho u_3,\rho E)^\top$, where ρ denotes the density, $u=(u_1,u_2,u_3)^\top$ is the velocity vector, and E is the specific total energy.

Harten [7] and Tadmor [8] showed that systems of conservation laws are symmetrizable if and only if they are equipped with a convex mathematical entropy function, $S(q)$. Given a set of conservation variables $q(x,t)$, the entropy variables which symmetrize the system are defined as the derivatives of the mathematical entropy function with respect to $q(x,t)$, $\partial S/\partial q$. Hughes and co-authors [9] extended these ideas to the compressible Navier–Stokes equations (1). Therein, it is shown that the mathematical entropy must be an affine function of the physical (or thermodynamic) entropy function and that semi-discrete solutions obtained from a weighted residual formulation based on entropy variables will respect the second law of thermodynamics. Hence, it is again found that the entropy function and the entropy variables are critical ingredients in the design of numerical schemes exhibiting non-linear stability.

In the specific case of the compressible Navier–Stokes equations (1), the entropy function is defined as $S=S(q)=-\rho s$, where s is the thermodynamic entropy. In the entropy analysis that will follow, the definition of the thermodynamic entropy for a perfect gas is the explicit form,

$$s=\frac{R}{\gamma-1}\log\left(\frac{T}{T_\infty}\right)-R\log\left(\frac{\rho}{\rho_\infty}\right), \tag{2}$$

where R, γ, T, T_∞, and ρ_∞ are the gas constant, the ratio of the heat capacity at constant pressure c_p to heat capacity at constant volume c_v, the temperature, and the reference temperature and density, respectively.

The scalar function $S=-\rho s$ satisfies the following conditions (see, for instance, [1,3] and the references therein):

- The function $S(q)$ when differentiated with respect to the conservative variables (i.e., $\partial S/\partial q$) simultaneously contracts all the inviscid spatial fluxes as follows

$$\frac{\partial S}{\partial q}\frac{\partial f_i^{(I)}}{\partial x_i}=\frac{\partial S}{\partial q}\frac{\partial f_i^{(I)}}{\partial q}\frac{\partial q}{\partial x_i}=\frac{\partial F_i}{\partial q}\frac{\partial q}{\partial x_i}=\frac{\partial F_i}{\partial x_i}, \quad i=1,2,3. \tag{3}$$

  The components of the contracting vector, $\partial S/\partial q$, are the entropy variables defined as $w^\top=\partial S/\partial q=\left(\frac{h}{T}-s-\frac{u_iu_i}{2T},\frac{u_1}{T},\frac{u_2}{T},\frac{u_3}{T},-\frac{1}{T}\right)$, where h denotes the specific entalphy which is defined as $h=c_pT$ for a perfect gas. $F_i(q)=-\rho u_i s$, $i=1,2,3$, are the entropy fluxes in the three Cartesian directions (see, for instance, [10]).
- The new entropy variables, w, symmetrize the system of equations (1):

$$\frac{\partial q}{\partial t}+\frac{\partial f_i^{(I)}}{\partial x_i}-\frac{\partial f_i^{(V)}}{\partial x_i}=\frac{\partial q}{\partial w}\frac{\partial w}{\partial t}+\frac{\partial f_i^{(I)}}{\partial w}\frac{\partial w}{\partial x_i}-\frac{\partial}{\partial x_i}\left(\widehat{c}_{ij}\frac{\partial w}{\partial x_j}\right)=0, \quad i=1,2,3 \tag{4}$$

  with: $\partial q/\partial w=(\partial q/\partial w)^\top$, $\partial f_i^{(I)}/\partial w=\left(\partial f_i^{(I)}/\partial w\right)^\top$ and $\widehat{c}_{ij}=\widehat{c}_{ij}^\top$. The matrices $\widehat{c}$ are positive semi-definite [11].
- The function $S(q)$ is convex, meaning that the Hessian, $\partial^2 S/\partial q^2=\partial w/\partial q$, is symmetric positive definite,

$$\zeta^T\frac{\partial^2 S}{\partial q^2}\zeta>0, \quad \forall\zeta\neq 0, \tag{5}$$

  and yields a one-to-one mapping from conservation variables, q, to entropy variables, $w^\top=\partial S/\partial q$. A sufficient (and also physical) condition to ensure the convexity of $S(q)$ is that $\rho,T>0$ [9,11].

[1] The symbol ∇q denotes the gradient of the conservative variables.

Remark 2.1. Dafermos [12] showed that if system of conservation laws is endowed with a convex entropy function, $S = S(q)$, a bound on the global estimate of $S = S(q)$ can be converted into an a priori estimate on the solution vector q (e.g., the solution of (1)). In fact, the convexity of the entropy function leads to the local well-posedness of the Cauchy problem in a Sobolev space of sufficiently high order, as well as the L^2-stability of the solution even within the broader class of entropy solution. The latter result is repeated here for completeness. Details of the proof can be found in Chapter 5 of Dafermos' book [12] and the references therein, and in the recent work of Svärd [13].

Given a C^3 entropy function $S(q)$ and sufficiently smooth initial data $q(x,0)$, the bound on the conservative variables at time t_f is related to the bound of the convex entropy function $S(q)$ in the following way

$$\int_\Omega q^\top(x,t_f)\,q(x,t_f)\,dx \le \frac{\mathscr{C}}{\left[\frac{\partial^2 S}{\partial q^2}(t_f)\right]_{min}} + \int_\Omega q(x,0)^\top q(x,0)\,dx, \tag{6}$$

where $\left[\frac{\partial^2 S}{\partial q^2}(t_f)\right]_{min} > 0$ is the minimum eigenvalue of the Hessian on the domain Ω (the Hessian is SPD, provided that $\rho, T > 0$), and $\mathscr{C}$ is an a priori known and bounded constant.

Contracting the system of Eqs. (1) with the entropy variables and using the relations given in (3) and (4) results in the differential form of the (scalar) entropy equation:

$$\begin{aligned}\frac{\partial S}{\partial q}\frac{\partial q}{\partial t} + \frac{\partial S}{\partial q}\frac{\partial f_i^{(I)}}{\partial x_i} = \frac{\partial S}{\partial t} + \frac{\partial F_i}{\partial x_i} = \frac{\partial S}{\partial q}\frac{\partial f_i^{(V)}}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(w^\top f_i^{(V)}\right) - \left(\frac{\partial w}{\partial x_i}\right)^\top f_i^{(V)} \\ &= \frac{\partial}{\partial x_i}\left(w^\top f_i^{(V)}\right) - \left(\frac{\partial w}{\partial x_i}\right)^\top \widehat{c}_{ij}\frac{\partial w}{\partial x_j}.\end{aligned} \tag{7}$$

Integrating Eq. (7) over the domain yields a global conservation statement for the entropy,

$$\frac{d}{dt}\int_\Omega S\,\mathrm{d}\mathbf{x} = \left[w^\top f_i^{(V)} - F_i\right]_{\partial\Omega} - \int_\Omega \left(\frac{\partial w}{\partial x_i}\right)^\top \widehat{c}_{ij}\frac{\partial w}{\partial x_j}\,\mathrm{d}\mathbf{x} = \left[w^\top f_i^{(V)} - F_i\right]_{\partial\Omega} - DT, \tag{8}$$

where DT is the viscous dissipation term,

$$DT = \int_\Omega \left(\frac{\partial w}{\partial x_i}\right)^\top \widehat{c}_{ij}\frac{\partial w}{\partial x_j}\,\mathrm{d}\mathbf{x} = \int_\Omega \begin{pmatrix}\partial w/\partial x_1\\ \partial w/\partial x_2\\ \partial w/\partial x_3\end{pmatrix}^\top \begin{pmatrix}\widehat{c}_{11} & \widehat{c}_{12} & \widehat{c}_{13}\\ \widehat{c}_{21} & \widehat{c}_{22} & \widehat{c}_{23}\\ \widehat{c}_{31} & \widehat{c}_{32} & \widehat{c}_{33}\end{pmatrix}\begin{pmatrix}\partial w/\partial x_1\\ \partial w/\partial x_2\\ \partial w/\partial x_3\end{pmatrix}\mathrm{d}\mathbf{x}. \tag{9}$$

The semi-discrete entropy estimate is obtained by mimicking term by term the continuous estimate given in Eq. (7). Herein, this is achieved by using an entropy stable discontinuous spectral collocation method of any order based on summation-by-parts (SBP) operators [1–3,5].

3. Summation-by-parts operators and semi-discretized system

First derivative operators that satisfy the SBP convention have the following form:

$$\begin{aligned}&\mathcal{D} = \mathcal{P}^{-1}\mathcal{Q}, \quad \mathcal{P} = \mathcal{P}^\top, \quad \boldsymbol{\zeta}^\top\mathcal{P}\boldsymbol{\zeta} > 0, \quad \boldsymbol{\zeta} \neq \mathbf{0},\\ &\mathcal{Q}^\top = \mathcal{B} - \mathcal{Q}, \quad \mathcal{B} = \mathrm{diag}\,(-1, 0, \ldots, 0, 1)\,.\end{aligned} \tag{10}$$

$\mathcal{D}$ is the discrete first derivative operator; $\mathcal{P}$ may be thought of as a mass matrix in the context of Galerkin finite-elements, and incorporates the local grid spacing into the derivative definition. Herein, the focus is exclusively on diagonal norm $\mathcal{P}$ operators. The nearly skew-symmetric matrix, $\mathcal{Q}$, is an undivided differencing operator (see, for instance, [1,5]).

All SBP derivative operators, $\mathcal{D}$, can be manipulated into the telescopic flux form [1,2],

$$\frac{df(\mathbf{q})}{dx} = \mathcal{P}^{-1}\mathcal{Q}\mathbf{f} + \mathcal{T}_{(p+1)} = \mathcal{P}^{-1}\Delta\bar{\mathbf{f}} + \mathcal{T}_{(p+1)}, \tag{11}$$

where $\mathcal{T}_{(p+1)}$ is the truncation error of the approximation and Δ is a non-square operator that calculates the undivided difference of the two adjacent fluxes at the flux points.[2] All the quantities located at flux points are denoted with an over-bar.

Consider a single tensor product element and an entropy stable spatially discontinuous collocation discretization with $N = p + 1$ solution points in each coordinate direction [1–3,5]; the following element-wise matrices will be used:

[2] For a high-order formulation like the nodal discontinuous Galerkin method, the flux points are a set of intermediate points prescribing bounding control volumes about each solution point. These points are similar in nature to the control volume edges employed in the finite volume method.

$$
\begin{aligned}
&\mathcal{D}_{x_1} = (\mathcal{D}_N \otimes I_N \otimes I_N \otimes I_5), \quad \cdots \quad \mathcal{D}_{x_3} = (I_N \otimes I_N \otimes \mathcal{D}_N \otimes I_5),\\
&\mathcal{P}_{x_1} = (\mathcal{P}_N \otimes I_N \otimes I_N \otimes I_5), \quad \cdots \quad \mathcal{P}_{x_3} = (I_N \otimes I_N \otimes \mathcal{P}_N \otimes I_5),\\
&\mathcal{P}_{x_1x_2} = (\mathcal{P}_N \otimes \mathcal{P}_N \otimes I_N \otimes I_5), \quad \cdots \quad \mathcal{P}_{x_2x_3} = (I_N \otimes \mathcal{P}_N \otimes \mathcal{P}_N \otimes I_5),\\
&\boldsymbol{\mathcal{P}} = \mathcal{P}_{x_1x_2x_3} = (\mathcal{P}_N \otimes \mathcal{P}_N \otimes \mathcal{P}_N \otimes I_5),\\
&\mathcal{B}_{x_1} = (\mathcal{B}_N \otimes I_N \otimes I_N \otimes I_5), \quad \cdots \quad \mathcal{B}_{x_3} = (I_N \otimes I_N \otimes \mathcal{B}_N \otimes I_5),\\
&\Delta_{x_1} = (\Delta_N \otimes I_N \otimes I_N \otimes I_5), \quad \cdots \quad \Delta_{x_3} = (I_N \otimes I_N \otimes \Delta_N \otimes I_5),
\end{aligned} \tag{12}
$$

where $\mathcal{D}_N$, $\mathcal{P}_N$, Δ_N, and $\mathcal{B}_N$ are the one-dimensional (1D) SBP operators [3], and I_N is the identity matrix of dimension N. I_5 denotes the identity matrix of dimension five.[3] The subscripts in (12) indicate the coordinate directions to which the operators apply (e.g., $\mathcal{D}_{x_1}$ is the differentiation matrix in the x_1 direction). The symbol $\otimes$ represents the Kronecker product. When applying these operators to the scalar entropy equation in space, a hat will be used to differentiate the scalar operator from the full vector operator. For example,

$$
\widehat{\boldsymbol{\mathcal{P}}} = (\mathcal{P}_N \otimes \mathcal{P}_N \otimes \mathcal{P}_N). \tag{13}
$$

Consider two cubic tensor product elements. Without loss of generality assume that all their faces are orthogonal to the three coordinate directions and are not boundary faces, i.e., they are not part of the boundary surface $\partial\Omega$. Using the SBP operators (12), system (1) is discretized locally on both elements as [3]

$$
\frac{\partial \mathbf{q}_l}{\partial t} + \mathcal{P}_{x_i,l}^{-1}\,\Delta_{x_i,l}\,\bar{\mathbf{f}}_{i,l}^{(I)} - \mathcal{D}_{x_i,l}\,\bar{\mathbf{f}}_{i,l}^{(V)} = \mathcal{P}_{x_i,l}^{-1}\,\mathbf{g}_{i,l}^{(In)}, \tag{14a}
$$

$$
\frac{\partial \mathbf{q}_r}{\partial t} + \mathcal{P}_{x_i,r}^{-1}\,\Delta_{x_i,r}\,\bar{\mathbf{f}}_{i,r}^{(I)} - \mathcal{D}_{x_i,r}\,\bar{\mathbf{f}}_{i,r}^{(V)} = \mathcal{P}_{x_i,r}^{-1}\,\mathbf{g}_{i,r}^{(In)}, \tag{14b}
$$

where the subscripts l and r denote the left and right elements. The penalty interface terms $\mathbf{g}_{i,l}^{(In)}$ and $\mathbf{g}_{i,r}^{(In)}$ with $i = 1, 2, 3$ are used to patch together neighboring elements (see Section 4). The vector of conservative variables of each element is ordered as

$$
\mathbf{q} = \left(q\left(x_{(1)(1)(1)}\right)^\top, q\left(x_{(1)(1)(2)}\right)^\top, \ldots, q\left(x_{(N)(N)(N)}\right)^\top \right) = \left(q_{(1)}{}^\top, q_{(2)}{}^\top, \ldots, q_{(N^3)}{}^\top \right), \tag{15}
$$

where the subscripts denote the ordering of the solution points in the coordinate directions. The derivatives appearing in the viscous fluxes are also computed using the operator $\mathcal{D}_{x_i}$, $i = 1, 2, 3$, defined in (12).

4. Provably entropy stable interior interface coupling

Using and LDG-type approach [14], expressions (14) can be conveniently re-written as [3]

$$
\frac{\partial \mathbf{q}_l}{\partial t} + \mathcal{P}_{x_i,l}^{-1}\,\Delta_{x_i,l}\,\bar{\mathbf{f}}_{i,l}^{(I)} - \mathcal{D}_{x_i,l}\,[\widehat{c}_{ij,l}]\,\mathbf{\Theta}_{j,l} = \mathcal{P}_{x_i,l}^{-1}\,\mathbf{g}_{i,l}^{(In),q}, \tag{16a}
$$

$$
\mathbf{\Theta}_{i,l} - \mathcal{D}_{x_i}\mathbf{w}_l = \mathcal{P}_{x_i,l}^{-1}\,\mathbf{g}_{i,l}^{(In),\Theta}, \tag{16b}
$$

$$
\frac{\partial \mathbf{q}_r}{\partial t} + \mathcal{P}_{x_i,r}^{-1}\,\Delta_{x_i,r}\,\bar{\mathbf{f}}_{i,r}^{(I)} - \mathcal{D}_{x_i,r}\,[\widehat{c}_{ij,r}]\,\mathbf{\Theta}_{j,r} = \mathcal{P}_{x_i,r}^{-1}\,\mathbf{g}_{i,r}^{(In),q}, \tag{16c}
$$

$$
\mathbf{\Theta}_{i,r} - \mathcal{D}_{x_i}\mathbf{w}_l = \mathcal{P}_{x_i,r}^{-1}\,\mathbf{g}_{i,r}^{(In),\Theta}, \tag{16d}
$$

where $\mathbf{\Theta}_{i,l}$ and $\mathbf{\Theta}_{i,r}$ are the vectors of the gradient of the entropy variables on the left and right elements in the i direction, whereas $\mathbf{g}_{i,(\cdot)}^{(In),q}$ and $\mathbf{g}_{i,(\cdot)}^{(In),\Theta}$ are the penalty interface terms on the conservative variable and the gradient of the entropy variable, respectively [3]. The contributions of the penalty terms on the gradient of the entropy variables are non-zero only in the normal direction to the interface. The matrices $[\widehat{c}_{ij,\cdot}]$ are block diagonal matrices with five-by-five blocks corresponding to the viscous coefficients of each solution point. Note that (16) is obtained by using $f_i^{(V)} = \widehat{c}_{ij}\,\frac{\partial w}{\partial x_j} = \widehat{c}_{ij}\,\Theta_j$.

To obtain an equation for the entropy of the system, we follow the entropy stability analysis presented in [1,3]. Therefore, multiplying the two discrete equations in the left element by $\mathbf{w}_l^\top \boldsymbol{\mathcal{P}}_l$ and $\left([\widehat{c}_{ij,l}]\mathbf{\Theta}_{j,l}\right)^\top \boldsymbol{\mathcal{P}}_l$, respectively, and the two discrete equations in the right element by $\mathbf{w}_r^\top \boldsymbol{\mathcal{P}}_r$ and $\left([\widehat{c}_{ij,r}]\mathbf{\Theta}_{j,r}\right)^\top \boldsymbol{\mathcal{P}}_r$, respectively, the expression for the time derivative of the entropy function S in each element is

[3] The 3D compressible Navier–Stokes equations form a system of five non-linear partial differential equations.

$$
\begin{aligned}
&\frac{d}{dt}\,\mathbf{1}^\top \widehat{\mathcal{P}}_l\,\mathbf{S}_l + 2\left\|\sqrt{[\widehat{c}_{ij,l}]}\,\boldsymbol{\Theta}_{j,l}\right\|^2_{\mathcal{P}_l} + \mathbf{1}^\top\left(\widehat{\mathcal{P}}_{x_2x_3,l}\,\widehat{\mathcal{B}}_{x_1,l}\bar{\mathbf{F}}_{1,l} + \widehat{\mathcal{P}}_{x_1x_3,l}\,\widehat{\mathcal{B}}_{x_2,l}\bar{\mathbf{F}}_{2,l} + \widehat{\mathcal{P}}_{x_1x_2,l}\,\widehat{\mathcal{B}}_{x_3,l}\bar{\mathbf{F}}_{3,l}\right)\\
&\quad = \mathbf{w}_l^\top\left(\mathcal{P}_{x_2x_3,l}\,\mathcal{B}_{x_1,l}\,[\widehat{c}_{1j,l}]\boldsymbol{\Theta}_{j,l} + \mathcal{P}_{x_1x_3,l}\,\mathcal{B}_{x_2,l}\,[\widehat{c}_{2j,l}]\boldsymbol{\Theta}_{j,l} + \mathcal{P}_{x_1x_2,l}\,\mathcal{B}_{x_3,l}\,[\widehat{c}_{3j,l}]\boldsymbol{\Theta}_{j,l}\right)\\
&\quad + \mathbf{w}_l^\top\left(\mathcal{P}_{x_2x_3,l}\,\mathbf{g}_{1,l}^{(In),q} + \mathcal{P}_{x_1x_3,l}\,\mathbf{g}_{2,l}^{(In),q} + \mathcal{P}_{x_1x_2,l}\,\mathbf{g}_{3,l}^{(In),q}\right)\\
&\quad + \left([\widehat{c}_{1j,l}]\boldsymbol{\Theta}_{j,l}\right)^\top \mathcal{P}_{x_2x_3,l}\,\mathbf{g}_{1,l}^{(In),\Theta} + \left([\widehat{c}_{2j,l}]\boldsymbol{\Theta}_{j,l}\right)^\top \mathcal{P}_{x_1x_3,l}\,\mathbf{g}_{2,l}^{(In),\Theta} + \left([\widehat{c}_{3j,l}]\boldsymbol{\Theta}_{j,l}\right)^\top \mathcal{P}_{x_1x_2,l}\,\mathbf{g}_{3,l}^{(In),\Theta},
\end{aligned}
\tag{17a}
$$

$$
\begin{aligned}
&\frac{d}{dt}\,\mathbf{1}^\top \widehat{\mathcal{P}}_r\,\mathbf{S}_r + 2\left\|\sqrt{[\widehat{c}_{ij,r}]}\,\boldsymbol{\Theta}_{j,r}\right\|^2_{\mathcal{P}_r} + \mathbf{1}^\top\left(\widehat{\mathcal{P}}_{x_2x_3,r}\,\widehat{\mathcal{B}}_{x_1,r}\bar{\mathbf{F}}_{1,r} + \widehat{\mathcal{P}}_{x_1x_3,r}\,\widehat{\mathcal{B}}_{x_2,r}\bar{\mathbf{F}}_{2,r} + \widehat{\mathcal{P}}_{x_1x_2,r}\,\widehat{\mathcal{B}}_{x_3,r}\bar{\mathbf{F}}_{3,r}\right)\\
&\quad = \mathbf{w}_r^\top\left(\mathcal{P}_{x_2x_3,r}\,\mathcal{B}_{x_1,r}\,[\widehat{c}_{1j,r}]\boldsymbol{\Theta}_{j,r} + \mathcal{P}_{x_1x_3,r}\,\mathcal{B}_{x_2,r}\,[\widehat{c}_{2j,r}]\boldsymbol{\Theta}_{j,r} + \mathcal{P}_{x_1x_2,r}\,\mathcal{B}_{x_3,r}\,[\widehat{c}_{3j,r}]\boldsymbol{\Theta}_{j,r}\right)\\
&\quad + \mathbf{w}_r^\top\left(\mathcal{P}_{x_2x_3,r}\,\mathbf{g}_{1,r}^{(In),q} + \mathcal{P}_{x_1x_3,r}\,\mathbf{g}_{2,r}^{(In),q} + \mathcal{P}_{x_1x_2,r}\,\mathbf{g}_{3,r}^{(In),q}\right)\\
&\quad + \left([\widehat{c}_{1j,r}]\boldsymbol{\Theta}_{j,r}\right)^\top \mathcal{P}_{x_2x_3,r}\,\mathbf{g}_{1,r}^{(In),\Theta} + \left([\widehat{c}_{2j,r}]\boldsymbol{\Theta}_{j,r}\right)^\top \mathcal{P}_{x_1x_3,r}\,\mathbf{g}_{2,r}^{(In),\Theta} + \left([\widehat{c}_{3j,r}]\boldsymbol{\Theta}_{j,r}\right)^\top \mathcal{P}_{x_1x_2,r}\,\mathbf{g}_{3,r}^{(In),\Theta},
\end{aligned}
\tag{17b}
$$

where the quantity $\mathbf{1}$ represents a vector with N elements, (i.e., $\mathbf{1} = (1, 1, \ldots, 1)^\top$).

To simplify the notation, assume that the interface between the two tensor product cells lies at $x_1 = 0$. We also assume that all the points that lie on the other faces of the two cubes are treated in an entropy stable fashion; their contribution can then be neglected without loss of generality. Then, for our analysis we can just focus on a pair of interface nodes at $x_1 = 0$. We then introduce the operators $\mathbf{e}^{(-)}$ and $\mathbf{e}^{(+)}$ which "extract" from the cell-wise solution, flux, and interface penalty vectors only the variables associated to these two points.[4] Therefore, Eqs. (17) reduce to

$$
\begin{aligned}
\frac{d}{dt}\mathbf{1}^\top\widehat{\mathcal{P}}_l\,\mathbf{S}_l + \mathbf{1}^\top\widehat{\mathcal{P}}_{x_2x_3,l}\,\bar{\mathbf{F}}_{1,l} + 2\left\|\sqrt{[\widehat{c}_{1j,l}]}\boldsymbol{\Theta}_{j,l}\right\|^2_{\mathcal{P}_l} &= \mathbf{w}_l^\top\mathcal{P}_{x_2x_3,l}\,[\widehat{c}_{1j,l}]\boldsymbol{\Theta}_{j,l}\,\mathbf{e}^{(-)} + \mathbf{w}_l^\top\mathcal{P}_{x_2x_3,l}\,\mathbf{g}_{1,l}^{(In),q}\,\mathbf{e}^{(-)}\\
&\quad + \left([\widehat{c}_{1j,l}]\boldsymbol{\Theta}_{j,l}\right)^\top\mathcal{P}_{x_2x_3,l}\,\mathbf{g}_{1,l}^{(In),\Theta}\,\mathbf{e}^{(-)},
\end{aligned}
\tag{18a}
$$

$$
\begin{aligned}
\frac{d}{dt}\mathbf{1}^\top\widehat{\mathcal{P}}_r\,\mathbf{S}_r - \mathbf{1}^\top\widehat{\mathcal{P}}_{x_2x_3,r}\,\bar{\mathbf{F}}_{1,r} + 2\left\|\sqrt{[\widehat{c}_{ij,r}]}\boldsymbol{\Theta}_{j,r}\right\|^2_{\mathcal{P}_r} &= -\mathbf{w}_r^\top\mathcal{P}_{x_2x_3,r}\,[\widehat{c}_{1j,r}]\boldsymbol{\Theta}_{j,r}\,\mathbf{e}^{(+)} + \mathbf{w}_r^\top\mathcal{P}_{x_2x_3,r}\,\mathbf{g}_{1,r}^{(In),q}\,\mathbf{e}^{(+)}\\
&\quad + \left([\widehat{c}_{1j,r}]\boldsymbol{\Theta}_{j,r}\right)^\top\mathcal{P}_{x_2x_3,r}\,\mathbf{g}_{1,r}^{(In),\Theta}\,\mathbf{e}^{(+)}.
\end{aligned}
\tag{18b}
$$

The interface penalty terms are constructed as follows:

$$
\begin{aligned}
\mathbf{g}_{1,l}^{(In),q} &= \left[+\bar{\mathbf{f}}_1^{(I)(-)} - \mathbf{f}^{ssr}\left(q_i^{(-)}, q_i^{(+)}\right)\right] + \left[-\frac{1}{2}(1+\alpha)\left(\left[\widehat{c}_{1,j}^{(-)}\right]\boldsymbol{\Theta}_j^{(-)} - \left[\widehat{c}_{1,j}^{(+)}\right]\boldsymbol{\Theta}_j^{(+)}\right)\right]\\
&\quad + \left[\frac{1}{2}[L]\left(\mathbf{w}^{(-)} - \mathbf{w}^{(+)}\right)\right],
\end{aligned}
\tag{19a}
$$

$$
\mathbf{g}_{1,l}^{(In),\Theta} = \left[-\frac{1}{2}(1-\alpha)\left(\mathbf{w}^{(-)} - \mathbf{w}^{(+)}\right)\right],
\tag{19b}
$$

$$
\begin{aligned}
\mathbf{g}_{1,r}^{(In),q} &= \left[-\bar{\mathbf{f}}_1^{(I)(+)} + \mathbf{f}^{ssr}\left(q_i^{(-)}, q_i^{(+)}\right)\right] + \left[+\frac{1}{2}(1-\alpha)\left(\left[\widehat{c}_{1,j}^{(+)}\right]\boldsymbol{\Theta}_j^{(+)} - \left[\widehat{c}_{1,j}^{(-)}\right]\boldsymbol{\Theta}_j^{(-)}\right)\right]\\
&\quad + \left[\frac{1}{2}[L]\left(\mathbf{w}^{(+)} - \mathbf{w}^{(-)}\right)\right],
\end{aligned}
\tag{19c}
$$

$$
\mathbf{g}_{1,r}^{(In),\Theta} = \left[+\frac{1}{2}(1+\alpha)\left(\mathbf{w}^{(+)} - \mathbf{w}^{(-)}\right)\right].
\tag{19d}
$$

The LDG penalty terms involve the coefficients $\frac{1}{2}(1 \pm \alpha)$ and act only in the normal direction to the face. The IP terms involve the block diagonal parameter matrix, $[L]$, with N^3 five-by-five blocks, L, which are left unspecified for the moment.[5]

Herein, the solution between adjoining elements is allowed to be discontinuous. An inviscid interface flux that preserves the entropy consistency of the interior high-order accurate spatial operators [1] on either side of the interface $\mathbf{f}^{ssr}\left(q_i^{(-)}, q_i^{(+)}\right)$ is constructed as

$$
\mathbf{f}^{ssr}\left(q_i^{(-)}, q_i^{(+)}\right) = \mathbf{f}^{sr}\left(q_i^{(-)}, q_i^{(+)}\right) + \Lambda\left(w_i^{(+)} - w_i^{(-)}\right),
\tag{20}
$$

[4] These two vectors are zero at all points except at the "$(-)$, $(+)$" interface points.

[5] $N^3 = (p+1)^3$ is the number of solution points within a three-dimensional tensor product cell.

where $\mathbf{f}^{sr}\left(q_i^{(-)}, q_i^{(+)}\right)$ is the entropy *conservative* inviscid interface flux of any order p [11,1–3,5], and Λ is a negative semi-definite interface matrix with zero or negative eigenvalues. The superscripts $(-)$ and $(+)$ denote the collocated values on the left and right side of the interface, respectively. The entropy stable flux $\mathbf{f}^{ssr}\left(q_i^{(-)}, q_i^{(+)}\right)$ is more dissipative than the entropy conservative inviscid flux $\mathbf{f}^{sr}\left(q_i^{(-)}, q_i^{(+)}\right)$, as can be easily verified by contracting $\mathbf{f}^{ssr}\left(q_i^{(-)}, q_i^{(+)}\right)$ against the entropy variables [1]. Note that in [15] grid interfaces for entropy stable finite difference schemes are studied and interface fluxes similar to (20) are proposed.

Substituting expressions (19) and (20) in (18) and summing all the contributions of the two elements results in

$$\frac{d}{dt}\mathbf{1}^{\top}\widehat{\mathcal{P}}_l\mathbf{S}_l + \frac{d}{dt}\mathbf{1}^{\top}\widehat{\mathcal{P}}_r\mathbf{S}_r + 2\left[\left\|\sqrt{[\widehat{c}_{ij,l}]}\boldsymbol{\Theta}_{j,l}\right\|^2_{\mathcal{P}_l} + \left\|\sqrt{[\widehat{c}_{ij,r}]}\boldsymbol{\Theta}_{j,r}\right\|^2_{\mathcal{P}_r}\right] = \boldsymbol{\Upsilon}^{(I)} + \boldsymbol{\Upsilon}^{(V)}, \tag{21}$$

where $\boldsymbol{\Upsilon}^{(I)}$ and $\boldsymbol{\Upsilon}^{(V)}$ are the inviscid and the viscous interface terms. At the two interface nodes, these terms are

$$\Upsilon^{(I)} = \left(w^{(+)} - w^{(-)}\right)^{\top}\mathbf{f}^{ssr}\left(q_i^{(-)}, q_i^{(+)}\right) - \left(\psi^{(+)} - \psi^{(-)}\right) = \left(w^{(+)} - w^{(-)}\right)^{\top}\Lambda\left(w^{(+)} - w^{(-)}\right), \tag{22}$$

$$\Upsilon^{(V)} = \left(w_i^{(-)} - w_i^{(+)}\right) L \left(w_i^{(-)} - w_i^{(+)}\right). \tag{23}$$

Clearly, the interface contributions are dissipative if both the five-by-five matrices Λ and L are negative semi-definite. The matrix Λ can be constructed using different approaches, e.g., using an upwind operator that dissipates each characteristic wave based on the magnitude of its eigenvalue:

$$\begin{aligned}
\mathbf{f}^{ssc}\left(q_i^{(-)}, q_i^{(+)}\right) &= \mathbf{f}^{sr}\left(q_i^{(-)}, q_i^{(+)}\right) + 1/2\,\mathcal{Y}|\lambda|\mathcal{Y}^{\top}\left(w_i^{(-)} - w_i^{(+)}\right), \\
\frac{\partial}{\partial q}\mathbf{f}(q) &= \mathcal{Y}\lambda\mathcal{Y}^{\top}, \\
\frac{\partial q}{\partial w} &= \mathcal{Y}\mathcal{Y}^{\top},
\end{aligned} \tag{24}$$

where λ and $\mathcal{Y}$ are the diagonal matrix of the eigenvalues and the matrix of the eigenvectors, respectively. Note that the relation $\frac{\partial q}{\partial w} = \mathcal{Y}\mathcal{Y}^T$ is achieved by an appropriate scaling of the rotation eigenvectors. In [1,5], the matrix $\frac{\partial q}{\partial w}$ is constructed using the scaled eigenvectors introduced by Merriam [16]. Such an approach allows to impose an artificial viscosity from the viewpoint of numerical satisfaction of the second law of thermodynamics.

We are then left with the viscous interface term, $\boldsymbol{\Upsilon}^{(V)}$. The parameter values $\alpha = 0$ and $\alpha = \pm 1$ yield a symmetric LDG and a "flip-flop" narrow stencil LDG penalty, respectively. An LDG value of $\alpha = 0$ produces a global discrete operator that has a neutrally damped spurious eigenmode. Herein, this mode is damped using the IP dissipation. For a Reynolds number that approaches ∞, we would like the five-by-five matrix L to go to zero so that only the inviscid entropy stable penalty contributions in (19) are recovered. To achieve that, the matrix L is constructed as

$$L = -\beta^{(In)}\frac{\widehat{c}_{11}^{(-)} + \widehat{c}_{11}^{(+)}}{2\left(\mathcal{P}_{x_1}\right)_{(1)(1)}}, \quad \beta^{(In)} > 0, \tag{25}$$

where $\widehat{c}_{11}^{(-)}$ and $\widehat{c}_{11}^{(+)}$ are the positive semi-definite viscous coefficient matrices at the left and the right side of the interface in the normal direction ($i = j = 1$). The coefficient $\beta^{(In)}$ can be used to modify the strength of the IP penalty term, although excessively large values of $\beta^{(In)}$ reduce the maximum stable time step. In [3,5,6], $\beta^{(In)}$ was chosen to be the maximum value for which the explicit stability constraint remains unaffected. The factor $\left(\mathcal{P}_{x_1}\right)_{(1)(1)}$ in the denominator represents the normal local grid spacing which is incorporated in the diagonal SBP operator $\mathcal{P}$.[6] This term is introduced to get the correct dimension, and, as for the standard IP finite element approach, it increases the strength of L with increased resolution.

5. Conclusions

A provably entropy stable coupling technique which scales with the inverse of the Reynolds number has been proposed to patch together interior element interfaces for the three-dimensional compressible Navier–Stokes equations. This approach uses the well-established local discontinuous Galerkin and interior penalty techniques. Furthermore, it is remarkably easy to implement and compatible with any diagonal norm SBP spatial operator, including finite element, finite volume, finite difference schemes and the class of high-order accurate methods which include the large family of discontinuous Galerkin discretizations and flux reconstruction schemes.

[6] Herein, $\left(\mathcal{P}_{x_i}\right)_{(1)(1)}$ with $i = 1$ appears in the definition of L because the interface between the two elements is orthogonal to the x_1 direction [3].

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